On Expected Gaussian Random Determinants

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Abstract

The expectations of random determinants whose entries are real-valued, identically distributed, mean zero, correlated Gaussian random variables are examined using the Kronecker tensor products and some combinatorial arguments. This result is used to derive the expected determinants of $X + B$ and $AX + X'B$.

Key words: Radom Determinant; Covariance Matrix; Permutation Matrix; Kronecker Product; Principal Minor; Random Matrix

1 Introduction

We will consider an $n \times n$ matrix $X = (x_{ij})$, where x_{ij} is a real-valued, identically distributed Gaussian random variable with the expectation $\mathbf{E}x_{ij} = 0$ for all i, j. The individual elements of the matrix are not required to be independent. We will call such matrix a mean zero *Gaussian random matrix* and its determinant a *Gaussian random determinant*, which will be denoted by $|X|$. We are interested in finding the expectation of the Gaussian random determinant $\mathbf{E}|X|$. When x_{ij} is independent identically distributed, the odd order moments of $|X|$, i.e. $\mathbf{E}|X|^{2k-1}$ $(k = 1, 2, \ldots)$, are equal to 0 since $|X|$ has a symmetrical distribution with respect to 0. The exact expression for the even order moments, i.e. $\mathbf{E}|X|^{2k}$, $(k = 1, 2, \ldots)$, is also well known in relation with the Wishart distribution [10, p. 85-108]. Although one can study the moments of the Gaussian random determinant through standard techniques of matrix variate normal distributions [2,10], the aim of this paper is to examine the the expected Gaussian random determinant whose entries are correlated via the Kronecker tensor products which will be used in representing the covariance structure of X.

When n is odd, the expected determinant $\mathbf{E}|X|$ equals to zero regardless of the covariance structure of X. When n is even, the expected determinant can be computed if there is a certain underlying symmetry in the covariance structure of X. Let us start with the following well known Lemma [8].

Lemma 1 *For mean zero Gaussian random variables $Z_1, \ldots, Z_{2m+1}$,*

$$\mathbf{E}[Z_1 Z_2 \cdots Z_{2m+1}] = 0,$$

$$\mathbf{E}[Z_1 Z_2 \cdots Z_{2m}] = \sum_{i \in Q_m} \mathbf{E}[Z_{i_1} Z_{i_2}] \cdots \mathbf{E}[Z_{i_{2m-1}} Z_{i_{2m}}],$$

where Q_m is the set of the $(2m)!/m!2^m$ different ways of grouping $2m$ distinct elements of $\{1, 2, \ldots, 2m\}$ into m distinct pairs $(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})$ and each element of Q_m is indexed by $i = \{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\}$.

Lemma 1 is a unique Gaussian property. For example,

$$\mathbf{E}[Z_1 Z_2 Z_3 Z_4] = \mathbf{E}[Z_1 Z_2]\mathbf{E}[Z_3 Z_4] + \mathbf{E}[Z_1 Z_3]\mathbf{E}[Z_2 Z_4] + \mathbf{E}[Z_1 Z_4]\mathbf{E}[Z_2 Z_3].$$

The determinant of the matrix $X = (x_{ij})$ can be expanded as

$$|X| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{1\sigma(1)} \ldots x_{n\sigma(n)},$$

where S_n is the set whose $n!$ elements are permutations of $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ is the sign function for the permutation σ. Then applying Lemma 1 to this expansion, we have the expansion for $\mathbf{E}|X|$ in terms of the pair-wise covariances $\mathbf{E}[x_{ij} x_{kl}]$ [1].

Lemma 2 *For an $n \times n$ mean zero Guassian random matrix X, For n odd, $\mathbf{E}|X| = 0$ and for $n = 2m$ even,*

$$\mathbf{E}|X| = \sum_{\sigma \in S_{2m}} \operatorname{sgn}(\sigma) \sum_{i \in Q_m} \mathbf{E}[x_{i_1\sigma(i_1)} x_{i_2\sigma(i_2)}] \cdots \mathbf{E}[x_{i_{2m-1}\sigma(i_{2m-1})} x_{i_{2m}\sigma(i_{2m})}].$$

Lemma 2 will be our main tool for computing $\mathbf{E}|X|$. Before we do any computation, let us introduce some known results on the Kronecker products and the vec operator which will be used in representing the covariance structures of random matrices.

2 Preliminaries

The covariance structure of a random matrix $X = (x_{ij})$ is somewhat difficult to represent. We need to know

$$\mathbf{cov}(x_{ij}, x_{kl}) = \mathbf{E}[x_{ij}x_{kl}] - \mathbf{E}x_{ij}\mathbf{E}x_{kl}$$

for all i, j, k, l which have 4 indices and can be represented in terms of 4-dimensional array or the 4th order tensor but by vectorizing the matrix, we may use the standard method of representing the covariance structure of random vectors by a covariance matrix. Let $\text{vec}X$ be a vector of size pq defined by stalking the columns of the $p \times q$ matrix X one underneath the other. If x_i is the ith column of X, then $\text{vec}X = (x_1', \ldots, x_q')'$.

The covariance matrix of a $p \times q$ random matrix X denoted by $\mathbf{cov}X$ shall be defined as the $pq \times pq$ covariance matrix of $\text{vec}X$:

$$\mathbf{cov}X \equiv \mathbf{cov}(\text{vec}X) = \mathbf{E}[\text{vec}X(\text{vec}X)'] - \mathbf{E}[\text{vec}X]\mathbf{E}[(\text{vec}X)'].$$

Following the convention of multivariate normal distributions, if the mean zero Gaussian random matrix X has the covariance matrix Σ not necessarily nonsingular, we shall denote $X \sim N(0, \Sigma)$. For example, if the components of $n \times n$ matrix X are independent and identically distributed as Gaussian with zero mean and unit variance, $X \sim N(0, I_{n^2})$. Some authors have used $\text{vec}(A')$ instead of $\text{vec}A$ in defining the covariance matrix of random matrices [10, p. 79]. The pair-wise covariance $\mathbf{E}[x_{ij}x_{kl}]$ is related to the covariance matrix $\mathbf{cov}X$ by the following Lemma via the Kronecker tensor product $\otimes$.

Lemma 3 *For an $n \times n$ mean zero random matrix $X = (x_{ij})$,*

$$\mathbf{cov}X = \sum_{i,j,k,l=1}^{n} \mathbf{E}[x_{ij}x_{kl}]U_{jl} \otimes U_{ik},$$

where U_{ij} is an $n \times n$ matrix whose ijth entry is 1 and whose remaining entries are 0.

Proof. Note that $\mathbf{cov}X = \mathbf{E}[\text{vec}X(\text{vec}X)'] = \sum_{j,l=1}^{n} U_{jl} \otimes \mathbf{E}[x_j x_l']$ and $\mathbf{E}[x_j x_l'] = \sum_{i,k=1}^{n} \mathbf{E}[x_{ij}x_{kl}]U_{ik}$. Combining the above two result proves the Lemma. $\square$

Hence, the covariance matrix of X is an $n \times n$ block matrix whose ijth sub-matrix is the cross-covariance matrix between ith and jth columns of X. Now we need to define two special matrices K_{pq} and L_{pq}.

For a $p \times q$ matrix X, $\text{vec}(X')$ can be obtained by permuting the elements of $\text{vec}X$. Then there exists a $pq \times pq$ orthogonal matrix K_{pq} called a *permutation matrix* [5] such that

$$\text{vec}(X') = K_{pq}\text{vec}X. \tag{1}$$

The permutation matrix K_{pq} has the following representation [4]:

$$K_{pq} = \sum_{\substack{i=1..p \\ j=1..q}} U_{ij} \otimes U'_{ij}, \tag{2}$$

where U_{ij} is an $p \times q$ matrix whose ijth entry is 1 and whose remaining entries are 0. We shall define a companion matrix L_{pq} of K_{pq} as an $p^2 \times q^2$ matrix given by

$$L_{pq} = \sum_{\substack{i=1..p \\ j=1..q}} U_{ij} \otimes U_{ij}.$$

Unlike the permutation matrix K_{pq}, the matrix L_{pq} has not been studied much. The matrix L_{pp} has the following properties:

$$L_{pp} = L_{pp} K_{pp} = K_{pp} L_{pp} = \frac{1}{p} L_{pp}^2.$$

Let e_i be the ith column of the $p \times p$ identity matrix I_p. Then L_{pp} can be represented in a different way [9].

$$L_{pp} = \sum_{i,j=1}^{p} (e_i e_j') \otimes (e_i e_j') = \sum_{i,j=1}^{p} (e_i \otimes e_i)(e_j' \otimes e_j') = \text{vec} I_p (\text{vec} I_p)'. \tag{3}$$

Example 4 *For $p \times p$ matrices A and B,*

$$\text{tr}(A)\text{tr}(B) = (\text{vec}A)' L_{pp} \text{vec}B. \tag{4}$$

To see this, use the identity $\text{tr}(X) = (\text{vec}I_p)'\text{vec}X = \text{vec}X(\text{vec}I_p)'$ and apply Equation (3). It is interesting to compare Equation (4) with the identity $\text{tr}(AB) = (\text{vec}A)' K_{pp} \text{vec}B$.

Lemma 5 *If $p \times q$ matrix $X \sim N(0, I_{pq})$ then for $s \times p$ matrix A and $q \times r$ matrix B,*

$$AXB \sim N(0, (B'B) \otimes (AA')) \tag{5}$$

Proof. Since $\text{vec}(AXB) = (B' \otimes A)\text{vec}X$ [4, Theorem 16.2.1], $\mathbf{cov}(AXB) = (B' \otimes A)\mathbf{cov}X(B' \otimes A)' = (B' \otimes A)(B \otimes A') = (B'B) \otimes (AA')$. $\square$

Lemma 6 *If $p \times p$ matrix $X \sim N(0, L_{pp})$, then for $s \times p$ matrix A and $p \times r$ matrix B,*

$$AXB \sim N(0, \text{vec}(AB) \otimes \text{vec}(AB)) \tag{6}$$

Proof. We have $\mathbf{cov}(AXB) = (B' \otimes A)\mathbf{cov}X(B' \otimes A)'$. Using the identity (3), $\mathbf{cov}(AXB) = \Big((B' \otimes A)\text{vec}I_p\Big)\Big((B' \otimes A)\text{vec}I_p\Big)' = \text{vec}(AB)\Big(\text{vec}(AB)\Big)'$. $\square$

Note that for an orthogonal matrix Q and $X \sim N(0, L_{pp})$, the above Lemma shows $Q'XQ \sim N(0, L_{pp})$.

3 3 Basic covariance structures

In this section, we will consider three specific covariance structures $\mathbf{E}[x_{ij}x_{kl}] = a_{ij}a_{kl}$ (Theorem 7), $\mathbf{E}[x_{ij}x_{kl}] = a_{il}a_{jk}$ (Theorem 8) and $\mathbf{E}[x_{ij}x_{kl}] = a_{ik}a_{jl}$ (Theorem 9). The results on these three basic types of covariance structures will be the basis of constructing more complex covariance structures.

Theorem 7 *For $2m \times 2m$ Gaussian random matrix $X \sim N(0, \text{vec}A(\text{vec}A)')$,*

$$\mathbf{E}|X| = \frac{(2m)!}{m!2^m}|A|.$$

Proof. Let a_i be the ith column of $A = (a_{ij})$ and e_i be the ith column of I_{2m}. Then

$$\text{vec}A(\text{vec}A)' = \Big(\sum_{j=1}^{2m} e_j \otimes a_j\Big)\Big(\sum_{l=1}^{2m} e_l' \otimes a_l'\Big) = \sum_{j,l=1}^{2m}(e_je_l') \otimes (a_ja_l'). \tag{7}$$

Substituting $a_j a_l' = \sum_{i,k=1}^{2m} a_{ij} a_{kl} U_{ik}$ into Equation (7) and applying Lemma 3, we get $\mathbf{E}[x_{ij}x_{kl}] = a_{ij}a_{kl}$. Now apply Lemma 2 directly.

$$\mathbf{E}|X| = \sum_{i \in Q_m} \sum_{\sigma \in S_{2m}} \operatorname{sgn}(\sigma) a_{i_1\sigma(i_1)} \dots a_{i_{2m}\sigma(i_{2m})}.$$

Note that $\{i_1, \dots, i_{2m}\} = \{1, 2, \dots, 2m\}$. Therefore, the inner summation is the determinant of A and there are $\frac{(2m)!}{m!2^m}$ such determinant. $\square$

When $A = I_{2m}$, we have $X \sim N(0, L_{2m2m})$ and $\mathbf{E}|X| = \frac{(2m)!}{m!2^m}$.

One might try to generalize Theorem 7 to $\mathbf{E}[x_{ij}x_{kl}] = a_{ij}b_{kl}$ or $\mathbf{cov}X = \mathrm{vec}A(\mathrm{vec}B)'$ but this case degenerates into $\mathbf{E}[x_{ij}x_{kl}] = a_{ij}a_{kl}$. To see this, note that $\mathbf{E}[x_{ij}x_{kl}] = \mathbf{E}[x_{kl}x_{ij}] = a_{ij}b_{kl} = a_{kl}b_{ij}$. Then a_{ij} and b_{ij} should satisfy $a_{ij} = cb_{ij}$ for some constant c and for all i, j.

The case when $\mathbf{E}[x_{ij}x_{kl}] = \epsilon_{ijkl} - \delta_{ij}\delta_{kl}$, where ϵ_{ijkl} is a symmetric function in i, j, k, l and δ_{ij} is the Kronecker's delta, is given in [1, Lemma 5.3.2].

Theorem 8 *For $2m \times 2m$ $X = (x_{ij})$ and symmetric $A = (a_{ij})$ with $\mathbf{E}[x_{ij}x_{kl}] = a_{il}a_{jk}$ for all i, j, k, l,*

$$\mathbf{E}|X| = (-1)^m \frac{(2m)!}{m!2^m} |A|.$$

Proof. The condition $A = A'$ is necessary. Too see this, note that $\mathbf{E}[x_{ij}x_{kl}] = \mathbf{E}[x_{kl}x_{ij}] = a_{il}a_{jk} = a_{kj}a_{li}$. By letting $j = k$, we get $a_{il} = a_{li}$ for all i, l. Then by interchanging the order of the summations in Lemma 2.,

$$\mathbf{E}|X| = \sum_{i \in Q_m} \sum_{\sigma \in S_{2m}} \operatorname{sgn}(\sigma) a_{i_1\sigma(i_2)} a_{i_2\sigma(i_1)} \dots a_{i_{2m-1}\sigma(i_{2m})} a_{i_{2m}\sigma(i_{2m-1})}. \quad (8)$$

There exists a permutation τ such that

$$\tau(i_1) = \sigma(i_2), \tau(i_2) = \sigma(i_1), \ldots, \tau(i_{2m-1}) = \sigma(i_{2m}), \tau(i_{2m}) = \sigma(i_{2m-1}).$$

Then

$$\sigma^{-1}\tau(i_1) = i_2, \sigma^{-1}\tau(i_2) = i_1, \ldots, \sigma^{-1}\tau(i_{2m-1}) = i_{2m}, \sigma^{-1}\tau(i_{2m}) = i_{2m-1}.$$

Note that $\sigma^{-1}\tau$ is the product of m odd permutations called *transposition* which interchanges two numbers and leaves the other numbers fixed. Hence $\mathrm{sgn}(\sigma^{-1}\tau) = (-1)^m$. Then by changing the index from σ to τ in Equation (8) with $\mathrm{sgn}(\sigma) = (-1)^m \mathrm{sgn}(\tau)$, we get

$$\mathbf{E}|X| = (-1)^m \sum_{i \in Q_m} \sum_{\tau \in S_{2m}} \mathrm{sgn}(\tau) a_{i_1\tau(i_1)} a_{i_2\tau(i_2)} \ldots a_{i_{2m}\tau(i_{2m})}.$$

The inner summation is the determinant of A and there are $\frac{(2m)!}{m!2^m}$ such determinant. $\square$

Suppose $X \sim N(0, A \otimes A)$ with $A = (a_{ij})$. Since covariance matrices are symmetric, $A \otimes A = (A \otimes A)' = A' \otimes A'$. Then (a_{ij}) should satisfy $a_{ij}a_{kl} = a_{ji}a_{lk}$ for all i, j, k, l. By letting $i = j$, $a_{kl} = a_{lk}$ for all l, k so A should be symmetric. Now let us find the pair-wise covariance $\mathbf{E}[x_{ij}x_{kl}]$ when $\mathbf{cov}X = A \otimes A$ and $A = A'$. Note that

$$\mathbf{cov}X = A \otimes A = (\sum_{j,l=1}^{n} a_{jl}U_{jl}) \otimes (\sum_{i,k=1}^{n} a_{ik}U_{ik}) = \sum_{i,j,k,l=1}^{n} a_{ik}a_{jl}U_{jl} \otimes U_{ik}.$$

Following Lemma 3, the covariance structure $\mathbf{cov}X = A \otimes A, A = A'$ is equivalent to $\mathbf{E}[x_{ij}x_{kl}] = a_{ik}a_{jl}$ and $a_{ij} = a_{ji}$ for all i, j, k, l. Then we have the following Theorem for the case $\mathbf{E}[x_{ij}x_{kl}] = a_{ik}a_{jl}$.

Theorem 9 *For $2m \times 2m$ Gaussian random matrix $X \sim N(0, A \otimes A)$ and a symmetric positive definite $m \times m$ matrix A, $\mathbf{E}|X| = 0$.*

Proof. Since A is symmetric positive definite, there exists $A^{-1/2}$. Then following the proof of Lemma 5,

$$\mathbf{cov}(A^{-1/2} X A^{-1/2}) = (A^{-1/2} \otimes A^{-1/2})(A \otimes A)(A^{-1/2} \otimes A^{-1/2}) = I_{n^2}.$$

Hence $Y = A^{-1/2} X A^{-1/2} \sim N(0, I_{n^2})$. Since the components of Y are all independent, trivially $\mathbf{E}|Y| = 0$. Then it follows $\mathbf{E}|X| = |A|\mathbf{E}|Y| = 0$. $\square$

4 The expected determinants of $X + B$ and $AX + X'B$

The results developed in previous sections can be applied to wide range of Gaussian random matrices with more complex covariance structures. Since a linear combination of Gaussian random variables is again Gaussian, $X + B$ and $AX + X'B$ will be also Gaussian random matrices if X is a Gaussian random matrix when A and B are constant matrices. In this section, we will examine the expected determinants of $X + B$ and $AX + X'B$.

Theorem 10 *Let $n = 2m$. For $n \times n$ matrix $X \sim N(0, I_{n^2} + K_{nn})$,*

$$\mathbf{E}|X| = (-1)^m \frac{(2m)!}{m! 2^m}.$$

Proof. Note that

$$I_{n^2} = I_n \otimes I_n = \sum_{i,j,j,l=1}^{n} \delta_{ik}\delta_{jl} U_{jl} \otimes U_{ik} \tag{9}$$

and from Equation (2), $K_{nn} = \sum_{j,l=1}^{n} U_{jl} \otimes U_{lj} = \sum_{i,j,k,l=1}^{n} \delta_{jk}\delta_{il} U_{jl} \otimes U_{ik}$. Then from Lemma 3,

$$\mathbf{E}[x_{ij}x_{kl}] = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}.$$

Now apply Lemma 2 directly.

$$\mathbf{E}[x_{i_j\sigma(i_j)}x_{i_{j+1}\sigma(i_{j+1})}] = \delta_{i_j,i_{j+1}}\delta_{\sigma(i_j),\sigma(i_{j+1})} + \delta_{i_j,\sigma(i_{j+1})}\delta_{i_{j+1},\sigma(i_j)}.$$

Since $i_j \neq i_{j+1}$, the first term vanishes. Then we can modify the covariance structure of X to $\mathbf{E}[x_{ij}x_{kl}] = \delta_{jk}\delta_{il}$ and still get the same expected determinant. Now apply Theorem 8 with $A = I_{2m}$. $\square$

Similarly we have the following.

Theorem 11 *Let $n = 2m$. For $n \times n$ matrix $X \sim N(0, A \otimes A + \text{vec}B(\text{vec}B)')$ and a symmetric positive definite $n \times n$ matrix A,*

$$\mathbf{E}|X| = \frac{(2m)!}{m!2^m}|B|.$$

Proof. Since A is symmetric positive definite, there exists $A^{-1/2}$. Let $Y = (y_{ij}) = A^{-1/2}XA^{-1/2}$. Note that $\mathbf{E}|X| = |A|\mathbf{E}|Y|$. Now find the pair-wise covariance $\mathbf{E}[y_{ij}y_{kl}]$ and apply Lemma 2. Following the proof of Theorem 9,

$$\mathbf{cov}(Y) = I_{n^2} + (A^{-1/2} \otimes A^{-1/2})(\text{vec}B(\text{vec}B)')(A^{-1/2} \otimes A^{-1/2}).$$

Since $\text{vec}(A^{-1/2}BA^{-1/2}) = (A^{-1/2} \otimes A^{-1/2})\text{vec}B$,

$$\mathbf{cov}(Y) = I_{n^2} + \text{vec}(A^{-1/2}BA^{-1/2})(\text{vec}(A^{-1/2}BA^{-1/2}))'. \tag{10}$$

Then $\mathbf{E}[y_{ij}y_{kl}] = \delta_{ik}\delta_{jl} + \dots$, where $\delta_{ik}\delta_{jl}$ corresponds to the first term I_{n^2} in Equation (10) and $\dots$ indicates the second term which we do not compute.

To apply Lemma 2, we need the pair-wise covariance $\mathbf{E}[y_{i_j\sigma(i_j)}y_{i_{j+1}\sigma(i_{j+1})}] = \delta_{i_ji_{j+1}}\delta_{\sigma(i_j)\sigma(i_{j+1})} + \ldots$. Since $i_j \neq i_{j+1}$, the first term vanishes. Therefore, the expectation $\mathbf{E}|Y|$ will not change even if we modify the covariance matrix from Equation (10) to $\mathbf{cov}(Y) = \text{vec}(A^{-1/2}BA^{-1/2})(\text{vec}(A^{-1/2}BA^{-1/2}))'$. Then by applying Theorem 7, $\mathbf{E}|Y| = \frac{(2m)!}{m!2^m}|A^{-1/2}BA^{-1/2}|$. □

By letting $A = B = I_n$, we get

Corollary 12 *Let $n = 2m$. For $n \times n$ matrix $X \sim N(0, I_{n^2} + L_{nn})$,*

$$\mathbf{E}|X| = \frac{(2m)!}{m!2^m}.$$

The following theorem is due to [11], where the covariance structure is slightly different.

Theorem 13 *For $n \times n$ matrix $X \sim N(0, L_{nn})$ and a constant symmetric $n \times n$ matrix B,*

$$\mathbf{E}|X + B| = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(2j)!}{2^j j!}|B|_{n-2j},$$

where $\lfloor \frac{n}{2} \rfloor$ is the smallest integer greater than $\frac{n}{2}$ and $|B|_j$ is the sum of $j \times j$ principal minors of B.

Proof. Let Q be an orthogonal matrix such that $Q'BQ = D$, where $D = Diag(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix of eigenvalues of B. Then $|X + B| = |Q'XQ + D|$. $|Q'XQ + D|$ can be expanded in the following way [4, p. 196]:

$$|Q'XQ + D| = \sum_{\{i_1,\ldots,i_r\}} \lambda_{i_1} \cdots \lambda_{i_r} |Q'XQ^{\{i_1,\ldots,i_r\}}|, \qquad (11)$$

where the summation is taken over all 2^n subsets of $\{1,\ldots,n\}$ and $Q'XQ^{\{i_1,\ldots,i_r\}}$ is the $(n-r)\times(n-r)$ principal submatrix of $Q'XQ$ obtained by striking out the $i_1,\ldots,i_r$th rows and columns. From Lemma 6, $Q'XQ \sim N(0, L_{nn})$. Then it follows the distribution of any $(n-r)\times(n-r)$ principal submatrix of $Q'XQ$ is $N(0, L_{(n-r)(n-r)})$. Using Theorem 7, $\mathbf{E}|Q'XQ^{\{i_1,\ldots,i_r\}}| = \frac{(2j)!}{j!2^j}$ for any $i_1,\ldots,i_r$ if $n-r=2j$. If $n-r=2j+1$, the principal minor equals 0. Therefore,

$$\mathbf{E}|X+B| = \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(2j)!}{j!2^j} \sum_{\{i_1,\ldots,i_r\}} \lambda_{i_1}\cdots\lambda_{i_r},$$

where the inner summation is taken over subsets of $\{1,\ldots,n\}$ with $r=n-2j$ fixed. The inner summation is called the rth *elementary symmetric function* of the n numbers $\lambda_1,\ldots,\lambda_n$ and it is identical to the sum of the $r\times r$ principal minors of B. [7, Theorem 1.2.12]. □

Theorem 14 *For $n\times n$ matrix $X \sim N(0, A\otimes A)$ and $n\times n$ symmetric positive definite A and symmetric $n\times n$ matrix B,*

$$\mathbf{E}|X+B| = |B|$$

Proof. Let $Y = A^{-1/2}XA^{-1/2}$. Then $Y \sim N(0, I_{n^2})$. Note that

$$\mathbf{E}|X+B| = |A|\mathbf{E}|Y + A^{-1/2}BA^{-1/2}|.$$

Following the proof of Theorem 13 closely,

$$\mathbf{E}|Y + A^{-1/2}BA^{-1/2}| = \sum_{i_1,\ldots,i_r} \lambda_{i_1}\cdots\lambda_{i_r}\mathbf{E}|Q'YQ^{\{i_1,\ldots,i_r\}}|,$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $A^{-1/2}BA^{-1/2}$. Since $Q'YQ^{\{i_1,\ldots,i_r\}} \sim N(0, I_{(n-r)^2})$, $\mathbf{E}|Q'YQ^{\{i_1,\ldots,i_r\}}| = 0$ if $r<n$ while

$$\mathbf{E}|Q'YQ^{\{i_1,i_2,\ldots,i_n\}}| = \mathbf{E}|Q'YQ^{\{1,2,\ldots,n\}}| = 1.$$

Hence $\mathbf{E}|Y + A^{-1/2}BA^{-1/2}| = \lambda_1 \cdots \lambda_n = |A^{-1}||B|$. $\square$

Theorem 15 *Let $n = 2m$. For $n \times n$ matrix $X \sim N(0, I_{n^2})$ and $n \times n$ constant matrix A and B,*

$$\mathbf{E}|AX + X'B| = (-1)^m m! |AB|_m,$$

where $|AB|_m$ is the sum of $m \times m$ principal minors of AB.

Proof. Let $Y = (y_{ij}) = AX + X'B$. we need to find the pair-wise covariance $\mathbf{E}[y_{ij}y_{kl}]$ using $\mathbf{E}[x_{ij}x_{kl}] = \delta_{ik}\delta_{jl}$. Note that $y_{ij} = \sum_u (a_{iu}x_{uj} + b_{uj}x_{ui})$. Let us use the Einstein convention of not writing down the summation $\sum_u$. We may write $y_{ij} \equiv a_{iu}x_{uj} + b_{uj}x_{ui}$. Then the pair-wise covariances of Y can be easily computed.

$$\begin{aligned}\mathbf{E}[y_{ij}y_{kl}] &= a_{iu}a_{kv}\mathbf{E}[x_{uj}x_{vl}] + a_{iu}b_{vl}\mathbf{E}[x_{uj}x_{vk}] + b_{uj}a_{kv}\mathbf{E}[x_{ui}x_{vl}] + b_{uj}b_{vl}\mathbf{E}[x_{ui}x_{vk}] \\ &= a_{iu}a_{ku}\delta_{jl} + a_{iu}b_{ul}\delta_{jk} + b_{uj}a_{ku}\delta_{il} + b_{uj}b_{ul}\delta_{ik}.\end{aligned}$$

Let $a_{(i)}$ be the ith row of A and b_i be the ith column of B respectively. Then $a_{iu}a_{ku} \equiv \sum_{u=1}^n a_{iu}a_{ku} = a'_{(i)}a_{(k)}$ and the other terms can be expressed similarly.

$$\mathbf{E}[y_{ij}y_{kl}] = a'_{(i)}a_{(k)}\delta_{jl} + a'_{(i)}b_l\delta_{jk} + a'_{(k)}b_j\delta_{il} + b'_j b_l \delta_{ik}. \tag{12}$$

When we apply Equation (12) to Lemma 2, the first and the last term vanish.

$$\mathbf{E}[y_{i_j\sigma(i_j)}y_{i_{j+1}\sigma(i_{j+1})}] = a'_{(i_j)}b_{\sigma(i_{j+1})}\delta_{i_{j+1}\sigma(i_j)} + a'_{(i_{j+1})}b_{\sigma(i_j)}\delta_{i_j\sigma(i_{j+1})}$$

Let τ be a permutation satisfying $\tau(i_1) = \sigma(i_2), \tau(i_2) = \sigma(i_1), \ldots, \tau(i_{2m-1}) = \sigma(i_{2m}), \tau(i_{2m}) = \sigma(i_{2m-1})$. Then

$$\mathbf{E}[y_{i_j\sigma(i_j)}y_{i_{j+1}\sigma(i_{j+1})}] = a'_{(i_j)}b_{\tau(i_j)}\delta_{i_{j+1}\tau(i_{j+1})} + a'_{(i_{j+1})}b_{\tau(i_{j+1})}\delta_{i_j\tau(i_j)}$$

and $\text{sgn}(\sigma) = (-1)^m \text{sgn}(\tau)$. By changing the summation index from σ to τ in Lemma 2,

$$\begin{aligned}\mathbf{E}|Y| = (-1)^m \sum_{\tau \in S_{2m}} \text{sgn}(\tau) \sum_{i \in Q_m} \Big(a'_{(i_1)} b_{\tau(i_1)} \delta_{i_2 \tau(i_2)} + a'_{(i_2)} b_{\tau(i_2)} \delta_{i_1 \tau(i_1)} \Big) \cdots \\ \Big(a'_{(i_{2m-1})} b_{\tau(i_{2m-1})} \delta_{i_{2m} \tau(i_{2m})} + a'_{(i_{2m})} b_{\tau(i_{2m})} \delta_{i_{2m-1} \tau(i_{2m-1})} \Big).\end{aligned}$$

The product term inside the inner summation can be expanded by

$$\sum_{\substack{j_1,\dots,j_m \\ k_1,\dots,k_m}} \delta_{j_1 \tau(j_1)} \cdots \delta_{j_m \tau(j_m)} a'_{k_1} b_{\tau(k_1)} \cdots a'_{k_m} b_{\tau(k_m)}, \tag{13}$$

where the summation is taken over 2^m possible ways of choosing $(j_l, k_l) \in \{(i_{2l-1}, i_{2l}), (i_{2l}, i_{2l-1})\}$ for all $l = 1, \dots, m$. In order to have a non-vanishing term in Equation (13), $\tau(j_1) = j_1, \dots, \tau(j_m) = j_m$. Let $\rho \in S'_m$ be a permutation of m numbers $\{k_1, \dots, k_m\}$. Then by changing the index from τ to ρ,

$$\begin{aligned}\mathbf{E}|Y| &= (-1)^m \sum_{i \in Q_m} \sum_{k_1,\dots,k_m} \sum_{\rho \in S'_m} \text{sgn}(\rho) a'_{(k_1)} b_{\rho(k_1)} \cdots a'_{(k_m)} b_{\rho(k_m)} \\ &= (-1)^m \sum_{i \in Q_m} \sum_{k_1,\dots,k_m} |AB_{\{k_1,\dots,k_m\}}|,\end{aligned}$$

where $AB_{\{k_1,\dots,k_m\}}$ is the $m \times m$ principal submatrix consisting of $k_1, \dots, k_m$th rows and columns of AB. Note that there are $\frac{(2m)!}{m! 2^m} \times 2^m = \frac{(2m)!}{m!}$ terms of principal minors $|AB_{\{k_1,\dots,k_m\}}|$ in the summation $\sum_{i \in Q_m} \sum_{k_1,\dots,k_m}$ but there are only $\binom{2m}{m}$ unique principal minors of AB. Then there must be repetitions of principal minors in the summation. Because of symmetry, the number of repetition for each principal minor must be $\frac{(2m)!}{m!} / \binom{2m}{m} = m!$. Hence $\sum_{i \in Q_m} \sum_{k_1,\dots,k_m} |AB_{\{k_1,\dots,k_m\}}| = m! |AB|_m$. $\square$

Corollary 16 *Let $n = 2m$. For $n \times n$ matrix $X \sim N(0, I_{n^2})$,*

$$\mathbf{E}|X + X'| = (-1)^m \frac{(2m)!}{m!}$$

Proof. Let $A = B = I_{2m}$ in Theorem 15. Use the fact that the sum of $m \times m$ principal minors of I_{2m} is $\binom{2m}{m}$. $\square$

Finally we propose a challenging problem. The difficulty of this problem arises from the restriction $m > n$.

Problem 17 *Let $m > n$. An $m \times n$ random matrix $X \sim N(0, A \otimes I_n)$, where the $m \times m$ matrix A is symmetric non-negative definite. For an $m \times m$ symmetric matrix C, determine $\mathbf{E}|X'CX|$.*

This problem was originally posed as Problem 23-1 in the Bulletin of the International Linear Algebra Society [3]. This is proved by Knight [6] using the elementary symmetric functions and induction on m starting from $m = n$. We will not duplicate the proof since Knight's solution is based on a different machiniary. Let ψ_j be the j-th elementary symmetric function of the eigenvalues of $\lambda_1, \cdots, \lambda_m$ of $A^{1/2}CA^{1/2}$. The j-th elementary symmetric function is the sum of all products of j distinct λ's. For example,

$$\begin{aligned}
\psi_1(\lambda_1, \cdots, \lambda_m) &= \lambda_1 + \cdots + \lambda_m, \\
\psi_2(\lambda_1, \cdots, \lambda_m) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{m-1}\lambda_m, \\
\psi_m(\lambda_1, \cdots, \lambda_m) &= \lambda_1\lambda_2 \cdots \lambda_m.
\end{aligned}$$

Then it can be shown that $\mathbf{E}|X'CX| = n!\psi_n(\lambda_1, \cdots, \lambda_m)$.

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